Report 2015

MATHEMATICAL ANALYSIS OF CERTAIN PHASES OF THE ARMY PHYSICAL SUPPLY DISTRIBUTION SYSTEM

by

Dr. B. D. Sivazlian

October 1971

Approved for public release; distribution unlimited.

U. S. ARMY MOBILITY EQUIPMENT RESEARCH AND DEVELOPMENT CENTER
FORT BELVOIR, VIRGINIA

53

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D		
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)		
1. ORIGINATING ACTIVITY (Corporate author) Mechanical Technology Department, U. S. Army Mobility Equipment Research and Development Center, Fort Belvoir, Virginia 22060	2a. REPORT SECURITY CLASSIFICATION Unclassified	
	2b. GROUP N/A	
3. REPORT TITLE MATHEMATICAL ANALYSIS OF CERTAIN PHASES OF THE ARMY PHYSICAL SUPPLY DISTRIBUTION SYSTEM		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Final Report June–September 1969		
5. AUTHOR(S) (First name, middle initial, last name) B. D. Sivazlian		
6. REPORT DATE October 1971	7a. TOTAL NO. OF PAGES 50	7b. NO. OF REFS 5
8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S) 2015	
b. PROJECT NO.		
c. Task: 1G662708D50701	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
d.		
10. DISTRIBUTION STATEMENT Approved for public release; distribution unlimited.		
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Mechanical Technology Department, USAMERDC, Fort Belvoir, Va. 22060	

13. ABSTRACT

This report is a mathematical analysis of selected phases of the Army Physical Supply Distribution System (APSDS). The first part is a stochastic analysis of convoy delays: it provides a statistical relation between distance travelled and time of travel and incorporates such parameters as frequency of stoppage, length of delay, and speed. The second part analyzes the problem of convoy attrition and losses under various hostile environments. Such relations as the expected number of units lost and the expected proportion of units surviving are obtained and expressed in terms of initial number of units in the convoy, travel time, and various interplaying combat intensity factors. Finally, a brief analysis of a particular terminal operation is given; expressions for cargo vessel turnaround times and handling cost per container are obtained.

DD FORM 1 NOV 65 1473 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE

UNCLASSIFIED
Security Classification

UNCLASSIFIED
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Containerization Unitization Mathematical Modeling Total System Concepts Physical Supply Distribution						

UNCLASSIFIED
Security Classification

1 -3-4- -38,114—Ft Belvoir

U. S. ARMY MOBILITY EQUIPMENT
RESEARCH AND DEVELOPMENT CENTER
FORT BELVOIR, VIRGINIA

Report 2015

MATHEMATICAL ANALYSIS OF CERTAIN PHASES OF THE ARMY PHYSICAL SUPPLY DISTRIBUTION SYSTEM

Task 1G662708D50701

October 1971

Distributed by

The Commanding Officer
U. S. Army Mobility Equipment Research and Development Center

Prepared by

Dr. B. D. Sivazlian
Mechanical Technology Department

Approved for public release; distribution unlimited.

SUMMARY

This report is a mathematical analysis of selected phases of the Army Physical Supply Distribution System (APSDS). The first part is a stochastic analysis of convoy delays; it provides a statistical relation between distance travelled and time of travel and incorporates such parameters as frequency of stoppage, length of delay, and speed. The second part analyzes the problem of convoy attrition and losses under various hostile environments. Such relations as the expected number of units lost and the expected proportion of units surviving are obtained and expressed in terms of initial number of units in the convoy, travel time, and various interplaying combat-intensity factors. Finally, a brief analysis of a particular terminal operation is given; expressions for cargo vessel turnaround times and handling cost per container are obtained.

FOREWORD

The present study was performed during the summer of 1969 (June 16–September 15) while the author was associated with the Research and Concepts Branch, Mechanical Technology Department, USAMERDC, Fort Belvoir, Virginia. The study was sponsored by AROD Contract DAH CO4 68C 0011.

The author is particularly grateful to Mr. D. P. Kane, Mr. E. C. Kinker, and Mr. E. J. Rodrick for the superb cooperation they provided him during his stay. The author acknowledges helpful discussions with the following persons during his research endeavor:

1. Messrs. E. Czul, D. P. Kane, J. K. Knael, E. J. Rodrick, and A. J. Rutherford, Mechanical Technology Dept., USAMERDC, Fort Belvoir, Va.

2. Major Houle, Materiel Branch, CDC Headquarters, Fort Belvoir, Va.

3. Major McDaniel, Division of Doctrines, CDC Headquarters, Fort Belvoir, Va.

4. Messrs. Cunningham, Smith, and Wood, Supply Agency, CDC, Fort Lee, Va.

5. Messrs. Good, Vrugtman, Whitney, and Wood, Transportation Agency, CDC, Fort Eustis, Va.

6. Mr. Goodman, MTMTS, Nassif Building, Alexandria, Va.

7. Captain Haynes and Mr. R. D. Rogers, U. S. Army Transportation School, Fort Eustis, Va.

This report was reviewed and finalized by Mr. J. C. Edmonds, Materials Handling Equipment Branch, Mechanical Technology Department.

CONTENTS

Section	Title	Page
	SUMMARY	ii
	FOREWORD	iii
	ILLUSTRATIONS	vi
I	INTRODUCTION	
	1. Scope	1
II	RETARDED CONVOY MOVEMENT	
	2. Introduction	2
	3. Dependent Analysis	3
	4. Expected Convoy Speed	4
	5. Space-Time Dependent Analysis	4
	a. Determination of $\overline{P}(t, x)$	4
	b. Determination of $\overline{\pi}(t, x)$–The Dual Problem	11
	6. Probability of Convoy Reaching Desingation on or before a Given Time, T	15
	7. Generalized Model for Convoy Retardation	15
	8. An Application to a Problem of Equipment Failure and Repair	17
III	CONVOY ATTRITION AND LOSSES	
	9. Introduction	17
	a. Model 1–Type 1 Attrition and Annihilation	19
	b. Model 2–Type 2 Attrition and Annihilation	20
	c. Model 3–Types 1 and 2 Attrition and Annihilation	21
	10. Cost Consideration in the Movement of a Convoy	23
IV	ANALYSIS OF A TERMINAL OPERATION– THE SEA-LAND CONTAINER TERMINAL SYSTEM	
	11. Introduction	23
	a. The Deterministic Model	24
	b. The Stochastic Model	32
	12. Effects of Crane Breakdown on Turnaround Time	36

CONTENTS (cont'd)

Section	Title	Page
V	CONCLUSIONS	
	13. Conclusions	36
	APPENDICES	
	A. Concept of Mobility of an Automotive Equipment Subject to Breakdown and Repairs	38
	B. Properties of the Functions f(N) and g(N)	39
	C. Analytic Properties of the Function C(n)	41

ILLUSTRATIONS

Figure	Title	Page
1	Space-Time Diagram for Convoy Movement Subject to Random Delay	5
2	Probability that at Time, t, a Distance of at least x Is Covered	12
3	Probability that a Distance, x, Will Be Covered on or Before Time, T	16
4	Gantt Chart for an Operation Consisting of Three Tractors	25
5	Waiting Time per Cycle as a Function of the Number of Tractors	27
6	Number of Containers Handled per Unit Time as a Function of the Number of Tractors	28
7	Total Ship Turnaround Time as a Function of the Number of Tractors (Deterministic Case)	30
8	Handling Cost per Container as a Function of the Number of Tractors Used (Deterministic Case)	31
9	Total Ship Turnaround Time as a Function of the Number of Tractors (Stochastic Case)	34
10	Expected Handling Cost per Container as a Function of the Number of Tractors Used (Stochastic Case)	35
11	Impact of Crane Breakdown upon the Ship Turnaround Time	37

MATHEMATICAL ANALYSIS OF CERTAIN PHASES OF THE

ARMY PHYSICAL SUPPLY DISTRIBUTION SYSTEM

I. INTRODUCTION

1. **Scope.** The present study is a mathematical analysis of selected phases of the Army Physical Supply Distribution System (APSDS) in order to attain a useful level of understanding before the necessary predictions can be made about overall system performance. In addition to providing basic approaches to formulate and solve various problems associated with the dynamics of the physical supply distribution system, the study demonstrates the feasibility of approaching the problem on a quantitative basis. As a result, it becomes possible to identify and correlate the relevant interplaying input parameters.

As reported in a previous study,[1] the APSDS can be conceptualized as a network system consisting of nodes and links – the nodes correspond to terminal points, and the links correspond to the movement of supply between two consecutive nodes. A logical procedure would be to study first each link and node and to combine eventually the results into a meaningful framework to arrive at optional solutions.

The link study is essentially an analysis of cargo retardation and attrition. Section II is devoted to a statistical analysis of convoy delays, and Section III analyzes the problem of convoy attrition and losses.

The operation at the nodes is more complex because it involves a larger array of variables and covers a wider spectrum of alternative systems. However, the cost and time associated with any terminal operation are two accepted important characteristics which need to be evaluated. To this end, Section IV illustrates a method of approaching the problem by analyzing the Sea-Land Container Terminal System; expressions for cargo vessel turnaround times and handling cost per container are obtained.

When the role of containers in the APSDS is evaluated, a fundamental problem area which cannot be overlooked relates to the extent of penetration of containers into forward areas. On the one hand, the utilization of larger containers presents economic advantages in the handling cost at nodes and in the transport cost between nodes; on

[1] B. D. Sivazlian, "A Preliminary Study Leading to a Mathematical Definition of an Optimum Future Army Physical Supply Distribution System," USAMERDC, Fort Belvoir, Virginia (August 1968).

the other hand, large containers could impede the mobility of the distribution system in forward areas and increase the vulnerability of convoys transporting the cargo. Therefore, one can legitimately wonder whether an optimum container dimension exists which would minimize the total cost of operation between two successive nodes. This problem has not been studied in the present report; however, the analytic results obtained to date provide the necessary ingredients to approach this important problem.

II. RETARDED CONVOY MOVEMENT

2. **Introduction.** During the normal course of its procession, a convoy may be immobilized at various time epochs, and each stoppage contributes to the delay of the convoy in reaching its final destination. Such retardation might be created by one or more natural or man-induced phenomena. As an example, an enemy attack might force a convoy to stop or to take evasive actions; the delay of the convoy is in general a function of the time length of the attack. Blockades resulting from barrier encounters are another cause for stoppage; here, the delay will be the time required to clear and remove the barriers before normal procession can resume.

Consider a convoy leaving at time $t = o$, origin A ($x = o$), with the objective of reaching destination B located at distance $x = X$. Let v be the normal convoy speed. We shall assume that: (1) events inducing convoy stoppage occur randomly in time according to a Poisson law with parameter λ (λ is the average number of convoy stoppages per unit time); (2) the delay associated with each stoppage is a random variable having an exponential distribution with parameter μ ($1/\mu$ is the average delay time per stoppage); and (3) the occurrence of each stoppage and the induced delays are independent events.

We shall derive an expression for the probability that at time, t, the convoy has covered at least a given distance and also an expression for the probability that a given distance will be covered on or before a specified time. The analysis proceeds in two steps: (1) a time-dependent model is developed; and (2) the results obtained are then used to develop a space-time dependent model involving both time and distance as variables.

At the outset, without any sophisticated analysis, it is possible to obtain an expression for the average convoy speed, $\bar{v}$, for long travel time (note that $\bar{v} < v$). If we denote by $\bar{x}$ any distance interval, then,

$$\bar{v} = \frac{\bar{x}}{\frac{\bar{x}}{v} + \text{expected delay}}$$

$$= \frac{\bar{x}}{\frac{\bar{x}}{v} + \frac{\bar{x}}{v} \cdot \frac{\lambda}{\mu}} = \frac{v}{1 + \lambda/\mu} .$$

The decrement ratio in speed is $\lambda / (\lambda + \mu)$. These results are no longer valid if the journey length of the convoy is short. It will be shown that the correct answers are functions of time.

3. **Dependent Analysis.** We define the following symbols:

$P_0(t)$ = probability that the convoy is stopped at time, t

$P_1(t)$ = probability that the convoy is moving at time, t

$\lambda\, dt$ = probability that the convoy will be stopped in the interval $(t, t + dt)$

$\mu\, dt$ = probability that the convoy will start moving in the time interval $(t, t + dt)$ given that it was at a stop at time, t

$0_i(dt)$ = function of dt such that $\lim_{dt \to 0} 0_i(dt)/dt = 0 \quad (i = 1,2)$

We then have the following relations:

$$P_o(t + dt) = P_o(t)(1 - \mu dt) + P_1(t)\,\lambda dt + 0_1(dt)$$

$$P_1(t + dt) = P_o(t)\,\mu dt + P_1(t)(1 - \lambda dt) + 0_2(dt) .$$

Carrying the usual operation and passing to the limit we obtain

$$\frac{dP_o(t)}{dt} = -\mu P_o(t) + \lambda P_1(t) , \qquad (1)$$

and $$\frac{dP_1(t)}{dt} = \mu P_o(t) - \lambda P_1(t) . \qquad (2)$$

It is evident that

$$P_o(t) + P_1(t) = 1 \qquad t \geqslant 0 . \qquad (3)$$

Initially, assume that the convoy has just started moving; then, $P_o(0) = 0$ and $P_1(0) = 1$. Using equations (1) and (3), we obtain the following differential equation for $P_o(t)$:

$$\frac{dP_o(t)}{dt} = -\mu P_o(t) + \lambda [1 - P_o(t)] \ .$$

Using the initial condition, we find as the solution of the above equation

$$P_o(t) = \frac{\lambda}{\lambda + \mu} \left[1 - e^{-(\lambda + \mu)t}\right] \tag{4}$$

Hence, $P_1(t) = 1 - \frac{\lambda}{\lambda + \mu} \left[1 - e^{-(\lambda + \mu)t}\right]$. (5)

4. **Expected Convoy Speed.** We define the expected instantaneous convoy speed as $\xi[v(t)]$, the expectation of $v(t)$. Now,

$$\xi[v(t)] = vP_1(t) = v - \frac{v\lambda}{\lambda + \mu} \left[1 - e^{-(\lambda + \mu)t}\right] \ ,$$

and in the steady state

$$\lim_{t \to \infty} \xi[v(t)] = \frac{v}{1 + \lambda/\mu} = \bar{v} \ .$$

5. **Space-Time Dependent Analysis.** In a space-time dependent analysis (Fig. 1), two state variables are introduced, namely, distance, x, and time, t. The determination of two basic probability distributions is of particular interest: the first is $\overline{P}(t,x)\,dx$, the probability that at time, t, the convoy will have travelled a distance between x and $x + dx$; the second is $\overline{\pi}(t,x)\,dt$, the probability that when the convoy is at a distance, x, the total elapsed time lies between t and $t + dt$.

a. **Determination of $\overline{P}(t, x)$.**

Define:

$P_o(t, x)\,dx$ = probability that the convoy is stopped at time, t, and lies between x and $x + dx$ $(0 < x < vt)$

$P_1(t, x)\,dx$ = probability that the convoy is moving at time, t, and lies between x and $x + dx$ $(0 < x < vt)$

with the symbols, λ, μ, and v, having the same interpretation as before. We shall assume that the normal convoy speed, v, is independent of time and distance travelled; thus $v = dx/dt$.

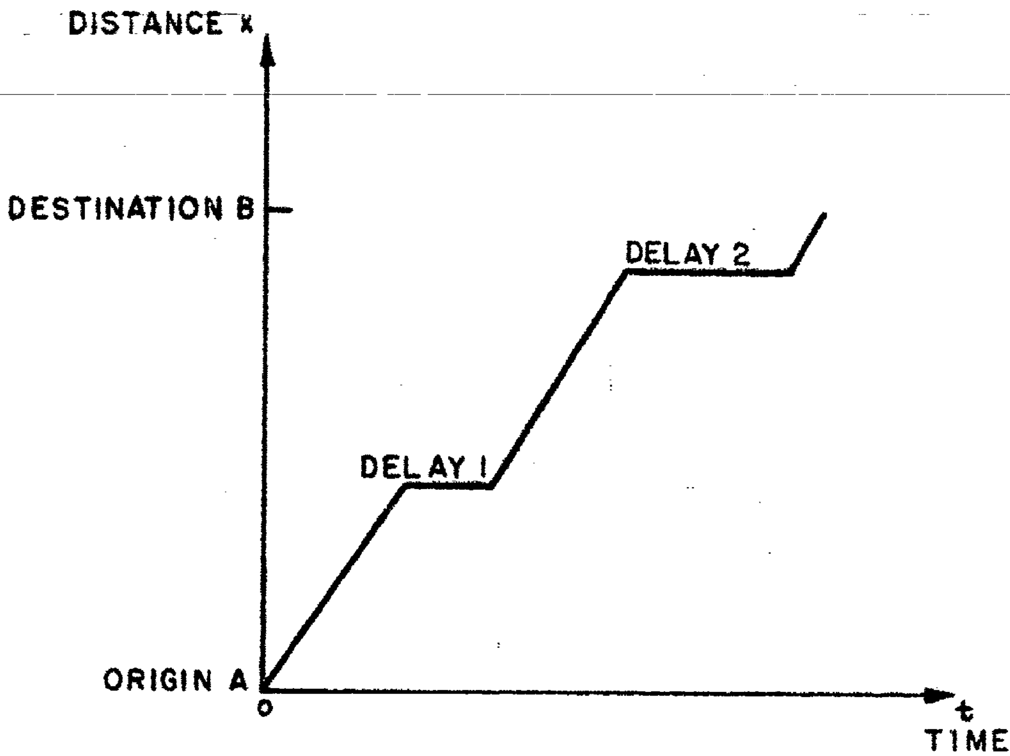


Fig. 1. Space-time diagram for convoy movement subject to random delay.

Upon enumerating the various probabilistic events, the following two relations in $P_o(t, x)$ and $P_1(t, x)$ are obtained:

$$P_o(t + dt, x)\, dx = P_o(t, x)\, dx\, (1 - \lambda\, dt)\, (1 - \mu\, dt)$$
$$+ P_o(t, x)\, dx\, \lambda\, dt(1 - \mu\, dt) + P_1(t, x)\, dx\, \lambda\, dt + 0_1(dt, dx) ; \quad (6)$$

and

$$P_1(t + dt, x)\, dx = P_o(t, x)\, dx\, (1 - \lambda dt)\, \mu dt + P_o(t, x)\, dx\, \lambda dt\, \mu dt$$
$$+ P_1(t, x - dx)\, dx\, (1 - \lambda dt) + 0_2(dt, dx), \quad (7)$$

where $\lim\limits_{dt \to 0} \dfrac{0_i(dt, dx)}{dt} = 0 \qquad i = 1,2,3,4.$

Relations (6) and (7) can be written, respectively, as:

$$\frac{P_o(t + dt, x) - P_o(t, x)}{dt} = -\mu P_o(t, x) + \lambda P_1(t, x) + \frac{0_3(dt, dx)}{dt} ;$$

and

$$\frac{P_1(t + dt, x) - P_1(t, x)}{dt} = \mu P_o(t, x) - \lambda P_1(t, x) - \frac{\partial P_1(t, x)}{\partial x} \frac{dx}{dt} + \frac{0_4(dt, dx)}{dt} .$$

Let $dt \to 0$; then, dropping for convenience the functional symbolism, we obtain for the two previous relations

$$\frac{\partial P_o}{\partial t} = -\mu P_o + \lambda P_1 \quad (8)$$

$$\frac{\partial P_1}{\partial t} = \mu P_o - \lambda P_1 - v \frac{\partial P_1}{\partial x} . \quad (9)$$

Let $P(t, x) = P_o(t, x) + P_1(t, x)$. (10)

Although any of the three quantities, P, P_o, P_1, can be determined first and the result together with equations (8), (9), or (10) can be used to derive the two remaining functions, it is easier to first evaluate the function $P_o(t,x)$.

Eliminating P_1 from (8) and (9), we obtain

$$\begin{vmatrix} \frac{\partial}{\partial t} + \mu & -\lambda \\ -\mu & \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \lambda \end{vmatrix} \cdot P_o = 0$$

or

$$\frac{\partial^2 P_o}{\partial x \partial t} + \frac{1}{v}\frac{\partial^2 P_o}{\partial t^2} + \frac{\lambda + \mu}{v}\frac{\partial P_o}{\partial t} + \mu\frac{\partial P_o}{\partial x} = 0. \tag{11}$$

Clearly, each of P and P_1 also satisfies equation (11). From the results of the time-dependent analysis, $P_o(x, t)$ satisfies the following boundary condition

$$\int_0^{vt} P_o(x\,t)\,dx = P_o(t) = \frac{\lambda}{\lambda + \mu}\left[1 - e^{-(\lambda+\mu)t}\right] . \tag{12}$$

Equation (11) is a linear partial differential equation of the hyperbolic type. The special boundary condition given by equation (12) does not allow one to obtain a solution using the method of separation of variables or by forming an appropriate Green's function. Using the transformation

$$w = x \qquad \text{and} \qquad z = vt - x,$$

equation (11) can be written in the canonical form

$$\frac{\partial^2 P_o}{\partial w \partial z} = -\frac{\mu}{v}\frac{\partial P_o}{\partial w} - \frac{\lambda}{v}\frac{\partial P_o}{\partial z} . \tag{13}$$

This equation is similar to one obtained by the author in a study of a periodic review multi-commodity inventory system.[2] Assume a solution of the form

$$P_o(w,z) = e^{-\frac{\lambda}{v}w - \frac{\mu}{v}z} \sum_{r=0}^{\infty} A_r w^r z^r . \tag{14}$$

Substituting equation (14) in (13), we obtain the following difference equation in A_r:

[2] B. D. Sivazlian, "Inventory Control of a Multi-Product System with Interacting Procurement," Working Paper (1966), Operations Research Group, Case Institute of Technology, Cleveland, Ohio.

$$r^2 A_r = \frac{\lambda\mu}{v^2} A_{r-1}$$

whose solution is

$$A_r = \left(\frac{\lambda\mu}{v^2}\right)^r \frac{1}{(r!)^2} A_o \tag{15}$$

with A_o being an arbitrary constant whose value can be obtained by using the boundary condition, equation (12). Substituting equation (15) in (14) we obtain

$$P_o(w, z) = A_o e^{-\frac{1}{v}(\lambda w + \mu z)} \sum_{r=0}^{\infty} \frac{\left(\frac{\lambda\mu}{v^2} wz\right)^r}{(r!)^2}$$

$$= A_o e^{-\frac{1}{v}(\lambda w + \mu z)} \sum_{r=0}^{\infty} \frac{\left(2\sqrt{\frac{\lambda\mu}{v^2} wz}\right)^{2r}}{2^{2r}(r!)^2}$$

Thus,

$$P_o(w, z) = A_o e^{-\frac{1}{v}(\lambda w + \mu z)} I_o\left(\frac{2}{v}\sqrt{\lambda\mu wz}\right)$$

where $I_o(\cdot)$ is the modified Bessel function of order zero. The expression for $P_o(x,t)$ is

$$P_o(x, t) = A_o e^{-\frac{1}{v}[\lambda x + \mu(vt - x)]} I_o\left[\frac{2}{v}\sqrt{\lambda\mu x(vt - x)}\right] \tag{16}$$

To determine the quantity, A_o, we have to evaluate the quantity

$$P_o(t) = A_o \int_0^{vt} e^{-\frac{1}{v}[\lambda x + \mu(vt - x)]} I_o\left[\frac{2}{v}\sqrt{\lambda\mu x(vt - x)}\right] dx.$$

Using the power series form for $I_o(\cdot)$, we obtain

$$P_o(t) = A_o \int_0^{vt} e^{-\frac{1}{v}[\lambda x + \mu(vt - x)]} \sum_{r=0}^{\infty} \frac{1}{(r!)^2} \left(\frac{\lambda x}{v}\right)^r \left[\frac{\mu}{v}(vt - x)\right]^r dx .$$

Since the infinite series is uniformly convergent, the interchange of the order of summation and integration becomes valid; hence

$$P_o(t) = A_o \sum_{r=0}^{\infty} \frac{1}{(r!)^2} \int_0^{vt} e^{-\frac{\lambda}{v}x} \left(\frac{\lambda x}{v}\right)^r \cdot e^{-\frac{\mu}{v}(vt - x)} \left[\frac{\mu}{v}(vt - x)\right]^r dx .$$

Let $\theta = vt$ and $P_o(t) = \tau(\theta)$ where

$$\tau(\theta) = A_o \sum_{r=0}^{\infty} \frac{1}{(r!)^2} \int_0^{\theta} e^{-\frac{\lambda}{v}x} \left(\frac{\lambda x}{v}\right)^r \cdot e^{-\frac{\mu}{v}(\theta - x)} \left[\frac{\mu}{v}(\theta - x)\right]^r dx . \qquad (17)$$

Define the Laplace transform of $\tau(\theta)$ as

$$L\left\{\tau(\theta)\right\} = \bar{\tau}(s) = \int_0^{\infty} e^{-s\theta} \tau(\theta)\, d\theta \qquad \mathrm{Re}\, s > 0.$$

Since the integral term under the summation sign in expression (17) is a convolution, we obtain, by taking the Laplace transform of both sides of (17),

$$\bar{\tau}(s) = A_o \sum_{r=0}^{\infty} \frac{1}{(r!)^2} L\left\{e^{-\frac{\lambda}{v}\theta}\left(\frac{\lambda\theta}{v}\right)^r\right\} \cdot L\left\{e^{-\frac{\mu}{v}\theta}\left(\frac{\mu\theta}{v}\right)^r\right\}$$

$$= A_o \sum_{r=0}^{\infty} \frac{\left(\frac{\lambda\mu}{v^2}\right)^r}{\left(s+\frac{\lambda}{v}\right)^{r+1}\left(s+\frac{\mu}{v}\right)^{r+1}}$$

$$= \frac{A_o}{\left(s+\frac{\lambda}{v}\right)\left(s+\frac{\mu}{v}\right)} \cdot \frac{1}{1 - \frac{\left(\frac{\lambda\mu}{v^2}\right)}{\left(s+\frac{\lambda}{v}\right)\left(s+\frac{\mu}{v}\right)}} \qquad \left|\left(s+\frac{\lambda}{v}\right)\left(s+\frac{\mu}{v}\right)\right| < 1$$

$$= \frac{A_o}{s\left(s+\frac{\lambda+\mu}{v}\right)}$$

$$= \frac{A_o v}{\lambda+\mu} \left[\frac{1}{s} - \frac{1}{s+\frac{\lambda+\mu}{v}}\right]$$

By a direct inversion, we obtain

$$\tau(\theta) = \frac{A_o v}{\lambda + \mu} \left[1 - e^{-\frac{\lambda + \mu}{v}\theta}\right]$$

$$P_0(t) = \frac{A_o v}{\lambda + \mu} \left[1 - e^{-(\lambda + \mu)t}\right]$$

Comparing this last expression with equation (12), we see that $A_o = \lambda/v$, hence

$$P_o(t, x) = \frac{\lambda}{v} e^{-\frac{\lambda x}{v} - \frac{\mu(vt - x)}{v}} I_o \left[2 \sqrt{\frac{\lambda x}{v} \cdot \frac{\mu(vt - x)}{v}}\right] \tag{18}$$

We next determine the quantity $P_1(t, x)$. From equation (8) we have

$$\frac{\partial}{\partial t}\left(e^{\mu t} P_o\right) = \lambda e^{\mu t} P_o$$

Hence

$$P_1(t,x) = \frac{1}{\lambda} e^{-\mu t} \frac{\partial}{\partial t}\left(e^{\mu t} P_o(t,x)\right)$$

$$= \frac{1}{v} e^{-\frac{\lambda x}{v} - \frac{\mu(vt - x)}{v}} \frac{\partial}{\partial t}\left\{I_o \left[2 \sqrt{\frac{\lambda x}{v} \cdot \frac{\mu(vt - x)}{v}}\right]\right\} .$$

A closed-form expression can be obtained in terms of $I_1(\cdot)$, the modified Bessel function of order 1:

$$P_1(t, x) = \frac{1}{v} e^{-\frac{\lambda x}{v} - \frac{\mu(vt - x)}{v}} \sqrt{\frac{\lambda x \mu}{vt - x}}\; I_1 \left[2 \sqrt{\frac{\lambda x}{v} \cdot \frac{\mu(vt - x)}{v}}\right].$$

Finally, using equation (10), we obtain for P(t,x)

$$P(t,x) = \frac{1}{v} e^{-\frac{\lambda x}{v} - \frac{\mu(vt - x)}{v}} \left\{\lambda I_o \left[2 \sqrt{\frac{\lambda x}{v} \cdot \frac{\mu(vt - x)}{v}}\right]\right.$$

$$\left. + \sqrt{\frac{\lambda x \mu}{vt - x}}\; I_1 \left[2 \sqrt{\frac{\lambda x}{v} \cdot \frac{\mu(vt - x)}{v}}\right]\right\} \quad 0 < x < vt . \tag{19}$$

In order to determine $\overline{P}(t,x)$, we note that the probability that there is no convoy stoppage over (0,t) is simply $e^{-\lambda t}$. This is also the probability that at time, t, the exact distance $x = vt$ will be travelled. Thus, $\overline{P}(t,x)$ has a mixed distribution with a finite probability concentration at $x = vt$ and a density function over $0 < x < vt$ defined by $P(t,x)$. We can thus write

$$\bar{P}(t,x) = \begin{cases} P(t,x) & 0 < x < vt \\ e^{-\lambda t} & x = vt \\ 0 & x > vt \end{cases} \tag{20}$$

The probability that at time, t, a distance of at least x be covered is

$$\int_0^x d\bar{P}(t,x)$$

whose configuration is depicted in Fig. 2.

b. Determination of $\bar{\pi}(t,x)$ — The Dual Problem. In the study of the dual problem, the quantity of interest is $\bar{\pi}(t,x)\,dt$ — the probability that when the convoy is at a distance, x, the total elapsed time lies between t and $t + dt$. The usage of the word "dual" will become apparent. It is clear that the structure of the function $\bar{\pi}(t,x)$ is such that

$$\bar{\pi}(t,x) = \begin{cases} 0 & t = 0 \\ e^{-\lambda \frac{x}{v}} & t = \frac{x}{v} \\ \pi(t,x) & t > \frac{x}{v} \end{cases} \tag{21}$$

since the elapsed time will be $t = \frac{x}{v}$ if the convoy travels a distance, x, at the normal speed without any stoppage, and the probability of such an event is $e^{-\lambda \frac{x}{v}}$. The function $\pi(t,x)$ is to be determined from probabilistic considerations.

Define $\pi_0(t,x)\,dt$ = the probability that when the convoy is at a distance, x, it is stopped and the total elapsed time lies between t and $t + dt$ $\quad (t > \frac{x}{v})$,

and $\pi_1(t,x)\,dt$ = the probability that when the convoy is at a distance, x, it is moving and the total elapsed time lies between t and $t + dt$ $\quad (t > \frac{x}{v})$.

It is clear that

$$\pi(t,x) = \pi_0(t,x) + \pi_1(t,x).$$

Basic probablistic arguments yield the two following relations in $\pi_0(t,x)$ and $\pi_1(t,x)$:

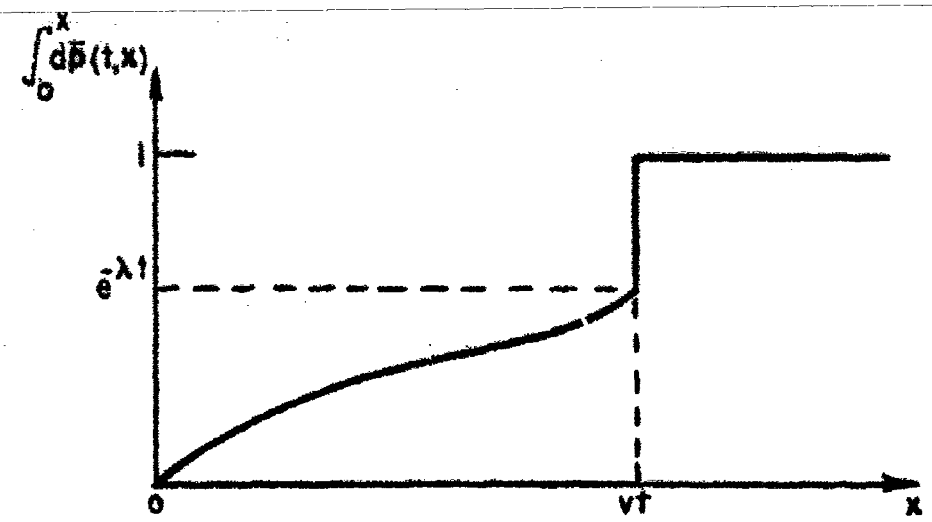


Fig. 2. Probability that at time, t, a distance of at least x is covered.

$$\pi_o(t,x)\,dt = \pi_o(t-dt, x)(1-\mu dt)\,dt + \pi_1(t-dt, x-dx)\lambda dt \cdot dt + \bar{o}_1(dt, dx)$$

and

$$\pi_1(t,x+dx)\,dt = \pi_1(t-dt, x)(1-\lambda dt)\,dt + \pi_o(t-dt, x)\mu dt + \bar{o}_2(dt, dx)$$

where

$$\lim_{dx \to 0} \frac{\bar{o}_i(dt, dx)}{dx} = 0 \qquad (i = 1,2).$$

Passing to the limit and dropping the functional notation, we obtain

$$\frac{\partial \pi_o}{\partial t} = -\mu\pi_o + \lambda\pi_1 \quad ,$$

and

$$v\frac{\partial \pi_1}{\partial x} + \frac{\partial \pi_1}{\partial t} = -\lambda\pi_1 + \mu\pi_o \ .$$

These equations are similar to equations (8) and (9); hence $\pi(t,x)$ satisfies the following equation:

$$\frac{\partial^2 \pi}{\partial x \partial t} + \frac{1}{v}\frac{\partial^2 \pi}{\partial t^2} + \frac{\lambda+\mu}{v}\frac{\partial \pi}{\partial t} + \mu\frac{\partial \pi}{\partial x} = 0. \tag{22}$$

The boundary condition for this partial differential equation is given by the normalizing condition

$$\int_{x/v}^{\infty} \pi(t,x)\,dt = 1 - e^{-\lambda\frac{x}{v}} . \tag{23}$$

Note that P(t,x) and $\pi(t,x)$ satisfy the same equation, hence the introduction of the notion of duality. On the basis of the results obtained in solving for P(t,x), one may assume that equation (22) has a solution of the form:

$$\begin{aligned}\pi(t,x) = {} & A\frac{\lambda}{v}\, e^{-\frac{\lambda x}{v} - \frac{\mu(vt-x)}{v}}\, I_o\left[2\sqrt{\frac{\lambda x}{v}\cdot\frac{\mu(vt-x)}{v}}\right] \\ & + B\cdot\frac{1}{v}\, e^{-\frac{\lambda x}{v} - \frac{\mu(vt-x)}{v}}\, \frac{\partial}{\partial t}\left\{I_o\left[2\sqrt{\frac{\lambda x}{v}\cdot\frac{\mu(vt-x)}{v}}\right]\right\}\end{aligned} \tag{24}$$

where A and B are arbitrary constants to be determined from the boundary condition of equation (23). Since the R.H.S. term in equation (24) needs to be integrated with respect to t over the range $\frac{x}{v} < t < \infty$, we evaluate first the following quantity:

$$\nu_1(x) = \int_{x/v}^{\infty} e^{-\frac{\lambda x}{v} - \frac{\mu(vt-x)}{v}} I_o\left[2\sqrt{\frac{\lambda x}{v} \cdot \frac{\mu(vt-x)}{v}}\right] dt$$

Let $z = vt - x$, then expressing $I_o(\cdot)$ as a power series and interchanging the order of integration and summation we obtain

$$\nu_1(x) = \sum_{r=0}^{\infty} \int_0^{\infty} e^{-\frac{\lambda x}{v} - \frac{\mu z}{v}} \frac{\left(\frac{\lambda x}{v}\right)^r \left(\frac{\mu z}{v}\right)^r}{(r!)^2} \cdot \frac{1}{v}\, dz$$

$$= \frac{1}{\mu} e^{-\frac{\lambda x}{v}} \sum_{r=0}^{\infty} \frac{\left(\frac{\lambda x}{v}\right)^r}{r!} \int_0^{\infty} \left(\frac{\mu z}{v}\right)^r \cdot \frac{1}{r!} \frac{\mu}{v} e^{-\frac{\mu}{v} z}\, dz = \frac{1}{\mu} \quad . \tag{25}$$

We next evaluate the quantity

$$\nu_2(x) = \int_{x/v}^{\infty} e^{-\frac{\lambda x}{v} - \frac{\mu(vt-x)}{v}} \frac{\partial}{\partial t}\left\{ I_o\left[2\sqrt{\frac{\lambda x}{v} \cdot \frac{\mu(vt-x)}{v}}\right]\right\} dt \ .$$

Integrating by parts, we obtain

$$\nu_2(x) = e^{-\frac{\lambda x}{v} - \frac{\mu(vt-x)}{v}} I_o\left[2\sqrt{\frac{\lambda x}{v} \cdot \frac{\mu(vt-x)}{v}}\right]\Bigg|_{\frac{x}{v}}^{\infty}$$

$$+ \mu \int_{\frac{x}{v}}^{\infty} e^{-\frac{\lambda x}{v} - \frac{\mu(vt-x)}{v}} I_o\left[2\sqrt{\frac{\lambda x}{v} \cdot \frac{\mu(vt-x)}{v}}\right] dt \ .$$

Using equation (25), we obtain

$$\nu_2(x) = 1 - e^{-\frac{\lambda x}{v}} \quad . \tag{26}$$

Using equations (24), (25), and (26) in (23) we obtain

$$\int_{\frac{x}{v}}^{\infty} \pi(t,x)\, dt = A\frac{\lambda}{v} \cdot \frac{1}{\mu} + \frac{B}{v}\left(1 - e^{-\frac{\lambda x}{v}}\right) = 1 - e^{-\lambda \frac{x}{v}} \quad .$$

Hence, $A = 0$ and $B = v$ and relation (24) can then be written as:

$$\pi(t,x) = e^{-\frac{\lambda x}{v} - \frac{\mu(vt-x)}{v}} \frac{\partial}{\partial t} \left\{ I_0 \left[2\sqrt{\frac{\lambda x}{v} \cdot \frac{\mu(vt-x)}{v}} \right] \right\}$$

$$= e^{-\frac{\lambda x}{v} - \frac{\mu(vt-x)}{v}} \cdot \frac{\frac{\lambda\mu}{v} x}{\sqrt{\frac{\lambda x}{v} \cdot \frac{\mu(vt-x)}{v}}} I_1 \left[2\sqrt{\frac{\lambda x}{v} \cdot \frac{\mu(vt-x)}{v}} \right]$$

$$t > \frac{x}{v} \qquad (27)$$

This last relation specifies completely the distribution function $\bar{\pi}(t,x)$ as defined in equation (21).

6. Probability of Convoy Reaching Destination on or before a Given Time, T. We are now in a position to evaluate the probability, M(T), that the convoy will reach its destination on or before a given time, T. Let x measure the distance from origin A to destination B.

Then

$$M(T) = \int_{-\infty}^{T} d\bar{\pi}(t,x) .$$

More explicitly, using the results of the previous analysis,

$$M(T) = \begin{cases} 0 & T < \frac{x}{v} \\ e^{-\frac{\lambda x}{v}} & T = \frac{x}{v} \\ e^{-\frac{\lambda x}{v}} + \int_{\frac{x}{v}}^{T} \pi(t,x)\, dt & T > \frac{x}{v} . \end{cases} \qquad (28)$$

The configuration of the function M(T) is as shown in Fig. 3. The expected time to travel a distance, X, is given by

$$\frac{X}{v} e^{-\frac{\lambda x}{v}} + \int_{x/v}^{\infty} T\,\pi(t,x)\, dt .$$

7. Generalized Model for Convoy Retardation. It is possible to formulate a general model for retarded convoy movement in which the parameters λ, μ, and v are

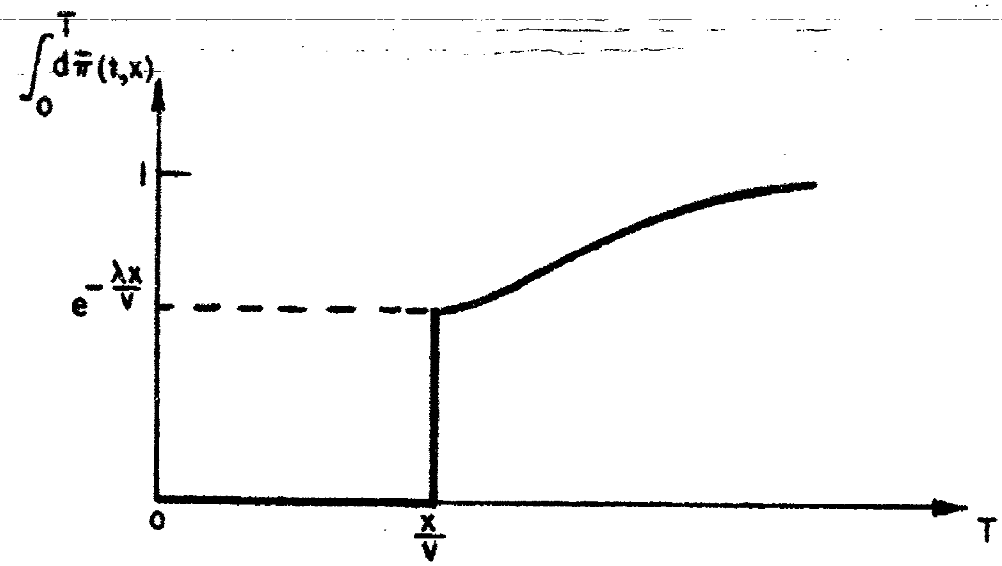


Fig. 3. Probability that a distance, x, will be covered on or before time, T.

functions of time, t, and the distance travelled, x. One can easily demonstrate, for example, that $P_0(t,x)$ and $P_1(t,x)$ still satisfy equations (8) and (9) with λ, μ, and v each a function of x and t. It can also be shown that $P_0(t,x)$, for example, satisfies the following hyperbolic partial differential equation:

$$\frac{\partial^2 P_o}{\partial x \partial t} + \frac{1}{v}\frac{\partial^2 P_o}{\partial t^2} + \mu \frac{\partial P_o}{\partial x} + f_1(t,x)\frac{\partial P_o}{\partial t} = f_2(t,x)\, P_o$$

where $f_1(t,x)$ and $f_2(t,x)$ are known functions of λ, μ, and v.

8. **An Application to a Problem of Equipment Failure and Repair.** The mathematical models discussed thus far find an interesting area of application in problems of failure and repair of a piece of equipment. In this case, the interplaying variables have the following interpretation:

t	=	time
x	=	number of hours of operations logged by the equipment
λ	=	average number of failures per unit time (failures are assumed to occur in a Poisson fashion)
$1/\mu$	=	average time of repair (the repair time is assumed to follow an exponential distribution)
v	=	unity

A specific application to this problem relates to the impact on cargo ship turnaround time of breakdowns of dock cranes during loading (unloading) operations. Further aspects of this problem will be discussed in Section IV.

Using the definition of M(T), an attempt is made in Appendix A to characterize quantitatively the concept of mobility of an automatic equipment subject to breakdowns and repairs.

III. CONVOY ATTRITION AND LOSSES

9. **Introduction.** Cargo shipments across a link between two given nodes take the form of convoys consisting of one or more vehicles carrying supplies from a given point of origin to a given destination. When the convoy system in a warfare environment is analyzed, it should normally be expected that, often, not all units of the convoy (and for that matter the convoy as a whole) will reach their destination: in general, enemy actions will tend to inhibit the normal supply operation and thus create delays, casualties, and losses.

In a generalized analysis of convoy movement, it is necessary to study the situation as a mixed-engagement problem involving four variables: time, t; the number of units in the convoy, n; the number of protective units escorting the convoy, k; and the number of attacking units, b.[3] By making suitable assumptions about the exchange rate between engaging forces, a Lanchester-type equation can be formulated for the probability that at a given time, t, the three state variables, n, k, and b, will take on particular values.

The complexity of this problem is increased by the number of interplaying state variables which makes doubtful the derivation of analytic results. The difficulty can be partly overcome by considering the number of cargo units to be the only significant variable and by assuming that the convoy proceeds in a dynamic environment (expressible as some function of time) causing attrition. To that end, the environment may be thought of as causing three types of losses:

a. Type 1 attrition induced by elemental attacks whose intensity is taken to be proportional to the number of units in the convoy. Thus, if there are n units in the convoy at time, t, the probability that a single unit will be lost in the time interval $(t, t + dt)$ is $n\lambda_1(t)\,dt$, $\lambda_1(t)$ being the intensity of attrition of Type 1 at time, t. In practice, a Type 1 attrition is usually caused by the lack of or the ineffectiveness of convoy protection.

b. Type 2 attrition induced by elemental attacks whose intensity is independent of the number of units in the convoy. Here, the probability that a single unit will be lost in the time interval $(t, t + dt)$ is $\lambda_2(t)dt$, $\lambda_2(t)$ being the intensity of attrition of Type 2 at time, t. Type 2 attrition results if convoys are effectively protected or if undetected mines are triggered.[4]

c. Total annihilation resulting from an environment affecting the totality of the convoy. We shall let $\mu(t)dt$ be the probability that in the time interval $(t, t + dt)$ the convoy is annihilated.

Three typical situations will be modeled. Model 1 assumes that convoy losses are due to Type 1 attrition and annihilation. Model 2 assumes that losses result from Type 2 attrition and annihilation. Finally, in Model 3, losses result from Type 1 and Type 2 attritions and total annihilation. For each of these models, an expression for P(n,t), the probability that at a given time, t, the number of units in the convoy is n, is derived. Using this expression, it becomes possible to assess the impact of the original convoy size, N, upon such quantities as the expected number of units reaching destination and the percentage amount of cargo lost.

[3] P. M. Morse and G. E. Kimball, *Methods of Operations Research*, J. Wiley, New York, 1951.

[4] *Ibid.*

a. **Model 1 – Type 1 Attrition and Annihilation.** The equations for $P(n,t)$ are

$$P(N, t+dt) = P(N,t)\,[1 - N\lambda_1(t)dt]\,[1-\mu(t)dt] \qquad n = N$$

$$P(n,t+dt) = P(n,t)\,[1 - n\lambda_1(t)dt]\,[1-\mu(t)dt]$$
$$+ P(n+1,t)\,(n+1)\,\lambda_1(t)\,dt\,[1-\mu(t)dt] \qquad n = 1,\ldots, N-1$$

$$P(0, t+dt) = P(0,t) + P(1,t)\,\lambda_1(t)dt\,[1-\mu(t)dt] + [1 - P(0,t)]\,\mu(t)\,dt \qquad n = 0$$

with the initial conditions $P(N,0) = 1$ and $P(n,0) = 0 \qquad n = 0, \ldots, N-1$

The above equations reduce to the following system of differential-difference equations:

$$\frac{dP(N,t)}{dt} = -\,[\mu(t) + N\lambda_1(t)]\;P(N,t) \qquad n = N$$

$$\frac{dP(n,t)}{dt} = -\,[\mu(t) + n\lambda_1(t)]\,P(n,t) + (n+1)\,\lambda_1(t)\,P(n+1,t) \qquad N = 1, \ldots N-1$$

$$\frac{dP(0,t)}{dt} = \mu(t)\,[1 - P(0,t)] + \lambda_1(t)\,P(1,t) \qquad n = 0$$

These equations can be solved recursively to yield

$$P(n,t) = \binom{N}{n} e^{-\int_0^t \mu(\theta)d\theta}\left[e^{-\int_0^t \lambda_1(\theta)d\theta}\right]^n \left[1 - e^{-\int_0^t \lambda_1(\theta)d\theta}\right]^{N-n} \qquad n = 1, 2, \ldots, N$$

$$P(0,t) = 1 - e^{-\int_0^t \mu(\theta)d\theta} + e^{-\int_0^t \mu(\theta)d\theta}\left[1 - e^{-\int_0^t \lambda_1(\theta)d\theta}\right]^N \qquad n = 0$$

The expected number of units reaching destination is

$$f(N) = \sum_{n=0}^{n=N} nP(n,t) = Ne^{-\int_0^t [\lambda_1(\theta) + \mu(\theta)]d\theta}$$

which is proportional to the number of units originally in the convoy. The expected number of convoy units lost is

$$N - f(N) = N\left[1 - e^{-\int_0^t [\lambda_1(\theta) + \mu(\theta)]\,d\theta}\right],$$

and the expected proportion loss in cargo is

$$g(N) = 1 - e^{-\int_0^t [\lambda_1(\theta) + \mu(\theta)] d\theta}$$

which is independent of the number of units in the convoy.

b. Model 2 — Type 2 Attrition and Annihilation. The equations for $P(n,t)$ are:

$$P(N,t+dt) = P(N,t) \quad [1 - \lambda_2(t)\,dt]\,[1 - \mu(t)\,dt] \qquad n = N$$

$$P(n,t+dt) = P(n,t) \quad [1 - \lambda_2(t)dt]\,[1 - \mu(t)\,dt]$$

$$+ \; P(n+1,t)\;\lambda_2(t)\,dt\,[1 - \mu(t)dt] \qquad n = 1, \ldots, N-1$$

$$P(0,t+dt) = P(0,t) + P(1,t)\,\lambda_2(t)\,dt\,[1 - \mu(t)\,dt]$$

$$+ \; [1 - P(0,t)]\;\mu(t)\,dt \qquad n = 0$$

with the initial conditions $P(N,0) = 1$ and $P(n,0) = 0$ $m = 0, 1 \ldots, N-1$. This reduces to the following system of differential difference equations:

$$\frac{dP(N,t)}{dt} = -[\lambda_2(t) + \mu(t)]\;P(N,t) \qquad n = N$$

$$\frac{dP(n,t)}{dt} = -[\lambda_2(t) + \mu(t)]\;P(n,t) + \lambda_2(t)\,P(n+1,t) \qquad n = 1,2,\ldots,N-1$$

$$\frac{dP(0,t)}{dt} = -\mu(t)\,P(0,t) + \mu(t) + \lambda_2(t)\,P(1,t) \qquad n = 0$$

These equations may be solved recursively to yield:

$$P(n,t) = \frac{\left[\int_0^t \lambda_2(\theta)d\theta\right]^{N-n}}{(N-n)!}\; e^{-\int_0^t [\lambda_2(\theta) + \mu(\theta)] d\theta} \qquad n = 1, 2, \ldots, N$$

$$P(0,t) = 1 - \sum_{M=1}^{M=N} P(n,t) \qquad n = 0$$

Let $$G(N) = \sum_{r=0}^{r=N} \frac{\left[\int_0^t \lambda_2(\theta)d\theta\right]^r}{r!}$$

The expected number of units reaching destination is

$$f(N) = \sum_{n=o}^{n=N} nP(n,t)$$

$$= \sum_{n=1}^{n=N} n \frac{\left[\int_o^t \lambda_2(\theta)d\theta\right]^r}{(N-n)!} e^{-\int_o^t (\lambda_2(\theta) + \mu(\theta)d\theta}$$

$$= e^{-\int_o^t (\lambda_2(\theta) + \mu(\theta)d\theta} \left[NG(N) - \int_o^t \lambda_2(\theta)d\theta \cdot G(N-1)\right].$$

In Appendix B, we show that the R.H.S. term is an increasing and convex function of N.

The expected proportion loss in cargo is

$$g(N) = 1 - e^{-\int_o^t (\lambda_2(\theta) + \mu(\theta)) d\theta} \left[G(N) - \frac{\int_o^t \lambda_2(\theta)\, d\theta}{N} G(N-1)\right].$$

We show in Appendix B that this is a decreasing function of N.

c. Model 3 – Types 1 and 2 Attrition and Annihilation. The equations for $P(n,t)$ are

$$P(N,t+dt) = P(N,t)\,[1 - N\lambda_1(t)dt]\,[1 - \lambda_2(t)dt]\,[1 - \mu(t)dt] \qquad n = N$$

$$P(n,t+dt) = P(n,t)\,[1 - n\lambda_1(t)dt]\,[1 - \lambda_2(t)dt]\,[1 - \mu(t)dt]$$
$$+ P(n+1,t)\,[n+1)\lambda_1(t)dt]\,[1 - \lambda_2(t)dt]\,[1 - \mu(t)dt]$$
$$+ P(n+1,t)\,[1 - (n+1)\lambda_1(t)dt]\,\lambda_2(t)dt\,[1 - \mu(t)dt] \qquad n = 1,2,\ldots, N-1$$

$$P(0,t+dt) = P(0,t) + P(1,t)\,\lambda_1(t)dt\,[1 - \mu(t)dt]\,[1 - \lambda_2(t)dt]$$
$$+ P(1,t)\,[1 - \lambda_1(t)dt]\,\lambda_2(t)dt\,[1 - \mu(t)dt] + [1 - P(0,t)]\mu(t)dt \qquad n = 0$$

The initial conditions are $P(N,0) = 1$ and $P(n,0) = 0$ $\qquad n = 1,2,\ldots, N-1$.
The corresponding differential-difference equations are:

$$\frac{dP(N,t)}{dt} = -\left[\mu(t) + \lambda_2(t) + N\lambda_1(t)\right] P(N,t) \qquad n = N$$

$$\frac{dP(n,t)}{dt} = -[\mu(t) + \lambda_2(t) + n\lambda_1(t)]\ P(n,t)$$

$$+\ [\lambda_2(t) + (n+1)\lambda_1(t)]\ P(n+1,t) \qquad n = 1, 2, \ldots, N-1$$

$$\frac{dP(0,t)}{dt} = -\mu(t)\ P(0,t) - \mu(t) + [\lambda_1(t) + \lambda_2(t)]\ P(1,t) \qquad n = 0$$

Let $$\varphi_1(t) = \int_0^t \lambda_1(u)\, e^{-\int_0^{\mu} \lambda_1(\theta)d\theta} du$$

$$\varphi_2(t) = \int_0^t \lambda_2(u)\, e^{-\int_0^{\mu} \lambda_1(\theta)d\theta} du$$

Then, we obtain recursively

$$P(N,t) = e^{-\int_0^t [\mu(\theta) + \lambda_2(\theta) + N\lambda_1(\theta)]\, d\theta}$$

$$P(N-1,t) = e^{-\int_0^t [\mu(\theta) + \lambda_2(\theta) + (N-1)\lambda_1(\theta)]\, d\theta} \quad [\varphi_2(t) + N\varphi_1(t)]$$

$$P(N-2,t) = e^{-\int_0^t [\mu(\theta) + \lambda_2(\theta) + (N-2)\lambda_1(\theta)]\, d\theta}$$

$$\int_0^t [\lambda_2(u) + (N-1)\lambda_1(u)]\ e^{-\int_0^{\mu} \lambda_1(\theta)d\theta}\ [\varphi_2(u) + N\varphi_1(u)]\, du$$

Unfortunately, closed-form expressions cannot be obtained for $P(n,t)$. For the case when $\lambda_1(t), \lambda_2(t)$, and $\mu(t)$ are independent of t, a simpler expression can be derived. Let $\alpha = \lambda_2/\lambda_1$, then

$$P(n,t) = e^{-(\lambda_2+\mu)t}\ \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+1)\,(N-n)!}\ (e^{-\lambda_1 t})^n\ (1 - e^{-\lambda_1 t})^{N-n} \qquad n = 1,2,\ldots,N$$

$$P(0,t) = 1 - e^{-\mu t} + \frac{\Gamma(N+\alpha+1)}{\Gamma(\alpha+1)\cdot\Gamma(N+1)}\ e^{-\mu t}\int_0^t e^{-\lambda_2\theta}\ d\left[\left(1 - e^{-\lambda_1\theta}\right)^N\right]$$

The expected number of units in the convoy is

$$f(N) = \sum_{n=1}^{n=N} n\, e^{-(\lambda_2+\mu)t}\ \frac{\Gamma(N+\alpha+1)}{\Gamma(n+\alpha+1)(N-n)!}\ \left(e^{-\lambda_1 t}\right)^n\ \left(1 - e^{-\lambda_1 t}\right)^{N-n}$$

A proof to demonstrate that $f(N)$ is an increasing function of N and that $g(N)$, the expected loss proportion in cargo, is a decreasing function of N eludes us for the present.

10. Cost Consideration in the Movement of a Convoy. The three basic cost elements that must be accounted for in the transport of cargo are:

a. The cost of transportation which is an increasing function of the number of units in the convoy, N.

b. The cost of providing a protective escort to the convoy; in general, it is expected that this cost be made up of two components, a fixed component and a variable component which are increasing functions of N.

c. The cost associated with the loss of cargo units; this, again, is expected to be, in general, an increasing function of N.

The configuration of the first two cost functions has to be determined empirically; although, as a first good approximation they can be assumed to be linear in N. The third cost function is easily determined from previous analyses since it will be equal to a constant time—the quantity $[N - f(N)]$. It then becomes possible to express the total cost of moving a convoy as a function of the number of units in the convoy. Finally, it might be a matter of interest to determine the cost of moving a single ton of commodity between given nodes.

IV. ANALYSIS OF A TERMINAL OPERATION — THE SEA-LAND CONTAINER TERMINAL SYSTEM

11. Introduction. A terminal, or node, can be defined as a transfer point where cargo materiel experiences one or more of the following operations: handling, storage, unitization, de-unitization. In general, the mathematical analysis at nodes will depend upon the particular terminal configuration and operation. In this chapter, the Sea-Land Container terminal operation is analyzed.

Containers (trailers) mounted on truck chassis wait in a marshalling area. When a container vessel docks, a stack of trailers in one of the vessel cells is unloaded one at a time onto waiting truck chassis by one or two gantry cranes. The chassis are pulled over to the marshalling area by tractor. Following this initial operation, the crane unloads an export container in the empty cell; and, on its return, the crane lifts an import container from an adjacent cell and deposits it on the truck chassis thus initiating a one-on, one-off cycle. Tractors, in the meantime, move back and forth between the dock and the marshalling areas pulling alternately import and export containers.

Because of the difference between the operating cycle times of crane and tractor, it becomes necessary to determine the optimum number of tractors in operation. Simulation studies of this problem were performed by the National Parts Council.[5] In this chapter, we develop two mathematical models to describe the problem. In the first model, the crane cycle time and the tractor cycle time are assumed to be deterministic variables; in the second model, the same variables are assumed to be stochastic.

Four basic operations to consider are:

- the tractor travel time
- the tractor waiting time
- the lift-on, lift-off crane operation time
- the crane waiting time

By minimizing the total cost of handling a unit container, the optimum number of tractors to be sequenced with the crane can be determined.

a. **The Deterministic Model.** Let:

n = number of tractors assigned to each crane
θ^c = operation cycle time of crane
θ^T = tractor travel time (assumed the same for all tractors)
w^c = waiting time of cranes
w^T = waiting time of a tractor
T = total cycle time
C_c = cost of operating the crane per unit time
C_T = cost of operating a tractor per unit time

Then
$$T = n\theta^c + w^c$$
$$= \theta^T + w^T + \theta^c$$

Hence
$$(n - 1)\theta^c + w^c = \theta^T + w^T .$$

It is evident that if the time parameters involved do not change over time, then either the crane waits or the tractors wait. These two cases are illustrated in Fig. 4.

[5] National Parts Council Research and Technical Bulletin No. 21967, 17 North Audley St., London, W. I.

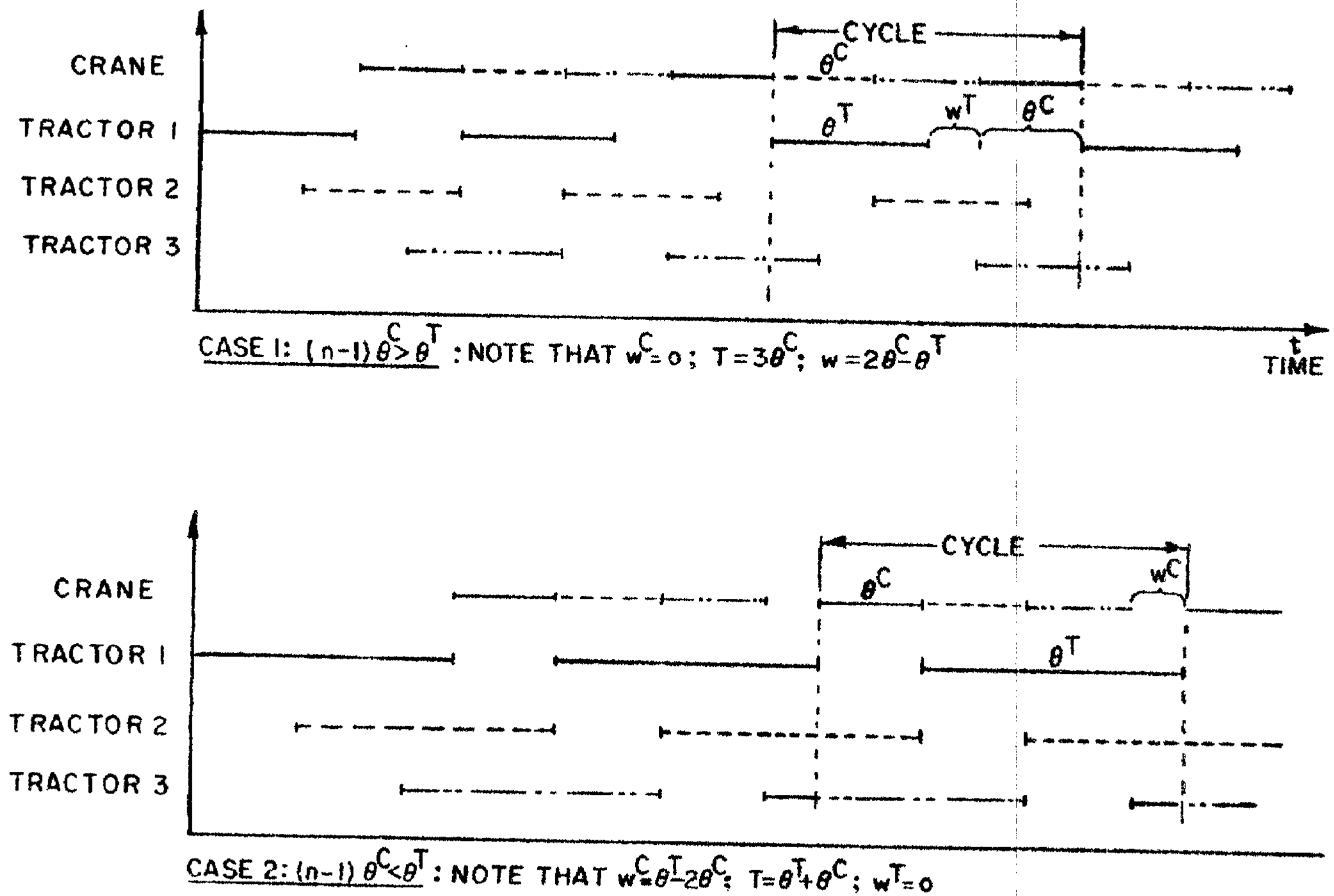


Fig. 4. Gantt chart for an operation consisting of three tractors.

(1) **Proportion of Waiting Time.** If $w^c = 0$, then $w^T \neq 0$, and $T = n\theta^c$ = cycle time. Each tractor will then wait an amount

$$w^T = (n-1)\theta^c - \theta^T \qquad\qquad (n-1)\theta^c > \theta^T$$

and the proportion of tractor idle time per cycle is

$$\frac{w^T}{T} = \frac{(n-1)\theta^c - \theta^T}{n\theta^c} \quad .$$

If now $w^c \neq 0$, then $w^T = 0$ and

$$T = \theta^T + \theta^c \quad .$$

The crane will then wait an amount equal to

$$w^c = \theta^T - (n-1)\theta^c \qquad\qquad \theta^T > (n-1)\theta^c$$

per cycle, and the proportion of crane waiting time is

$$\frac{w^c}{T} = \frac{\theta^T - (n-1)\theta^c}{\theta^T + \theta^c} \quad .$$

In Fig. 5, the waiting time per cycle is plotted as a function of the number of tractors in operation.

(2) **Loading – Unloading rate.** The loading – unloading rate is measured by the number of containers handled per unit time; the expression for this rate will depend on whether the crane waits or the tractors wait. It is given by the following:

$$\begin{cases} \dfrac{n}{\theta^T + \theta^c} & \text{if} \quad n \leqslant \dfrac{\theta^T}{\theta^c} + 1 \qquad \text{(crane waits)} \\[2ex] \dfrac{1}{\theta^c} & \text{if} \quad n \geqslant \dfrac{\theta^T}{\theta^c} + 1 \qquad \text{(tractor waits)} \end{cases}$$

Thus, if the crane is allowed to wait, the loading rate is proportional to the number of tractors involved; otherwise, if the tractors have to wait, the loading rate is constant. This is illustrated in Fig. 6.

(3) **Total Turnaround Time of Cargo Vessel.** Denote by N the total number of containers to be loaded (unloaded); then, neglecting the initial first cell clearing operation, the total turnaround time is given by

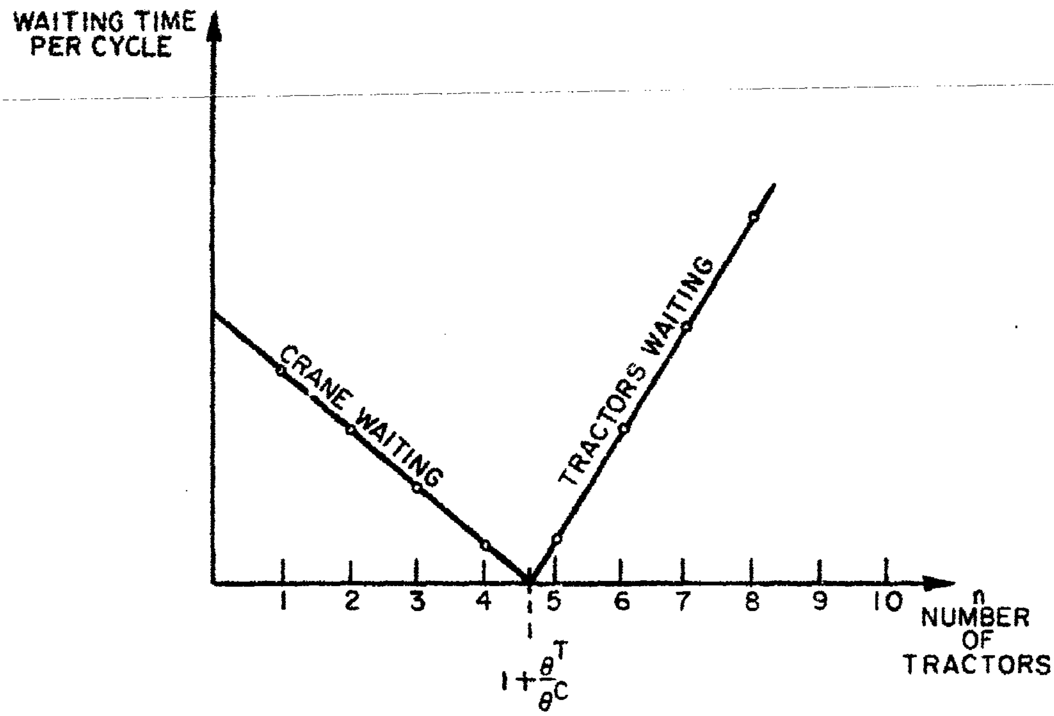


Fig. 5. Waiting time per cycle as a function of the number of tractors.

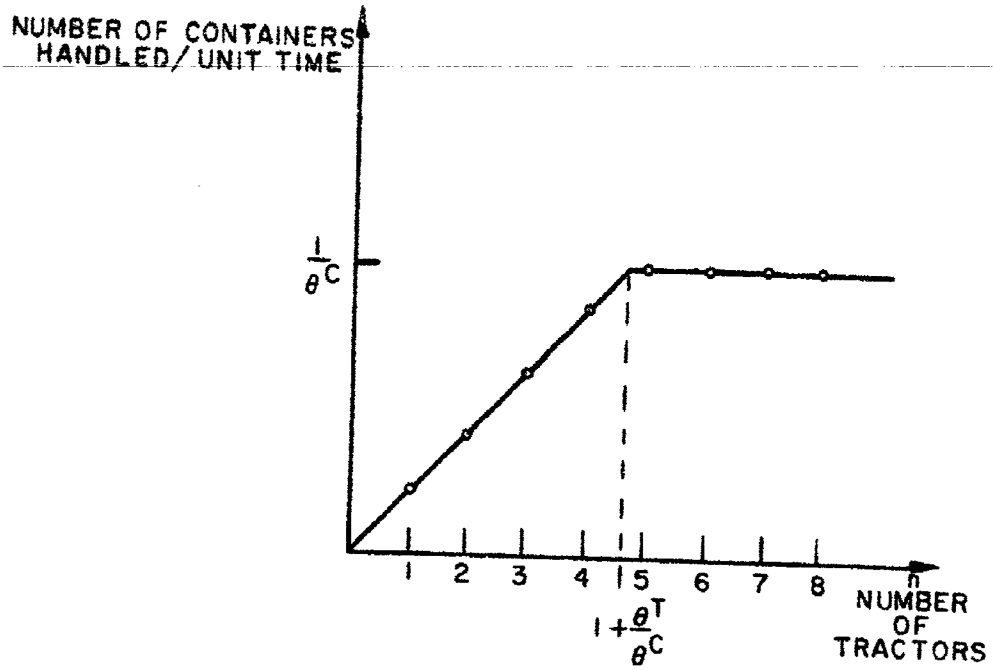


Fig. 6. Number of containers handled per unit time as a function of the number of tractors.

$$\begin{cases} \dfrac{N(\theta^T + \theta^c)}{n} & \text{if } n \leqslant 1 + \dfrac{\theta^T}{\theta^c} \\ N\theta^c & \text{if } n \geqslant 1 + \dfrac{\theta^T}{\theta^c} \end{cases}$$

and this is displayed as a function of the number of tractors in Fig. 7.

(4) **Optimum Number of Tractors in Operation.** Considering as our objective function the cost of handling a container, say $C(n)$, then

$$C(n) = \begin{cases} C_c \dfrac{\theta^T + \theta^c}{n} + C_T(\theta^T + \theta^c) & \text{if } (n-1) < \dfrac{\theta^T}{\theta^c} \\ C_c\,\theta^c + C_T\, n\theta^c & \text{if } (n-1) > \dfrac{\theta^T}{\theta^c} \end{cases}$$

The function $C(n)$ is shown in Fig. 8, and its analytic properties are investigated in Appendix C. There, it is shown that the optimum value of n, say n^*, which minimizes the function $C(n)$, is such that one of the following conditions are satisfied:

$$\left(\theta = \frac{\theta^T}{\theta^c}\ , \qquad C = \frac{C_c}{C_T}\ , \quad n^* \text{ integer}\right).$$

(a) Crane will wait:

$$\theta < \frac{-(C-\theta) + \sqrt{(C+\theta)^2 + 4C}}{2} < n^* < 1 + \theta$$

(b) Tractors wait:

$$(1+\theta) < n^* < 1 + \frac{-(C-\theta) + \sqrt{(C+\theta)^2 + 4C}}{2}$$

(c) No waiting for tractors or crane:

$$n^* = 1 + \theta$$

(d) Double solution (either crane waits or tractor waits):

$$n^* = \frac{-(C-\theta) + \sqrt{(C+\theta)^2 + 4C}}{2}$$

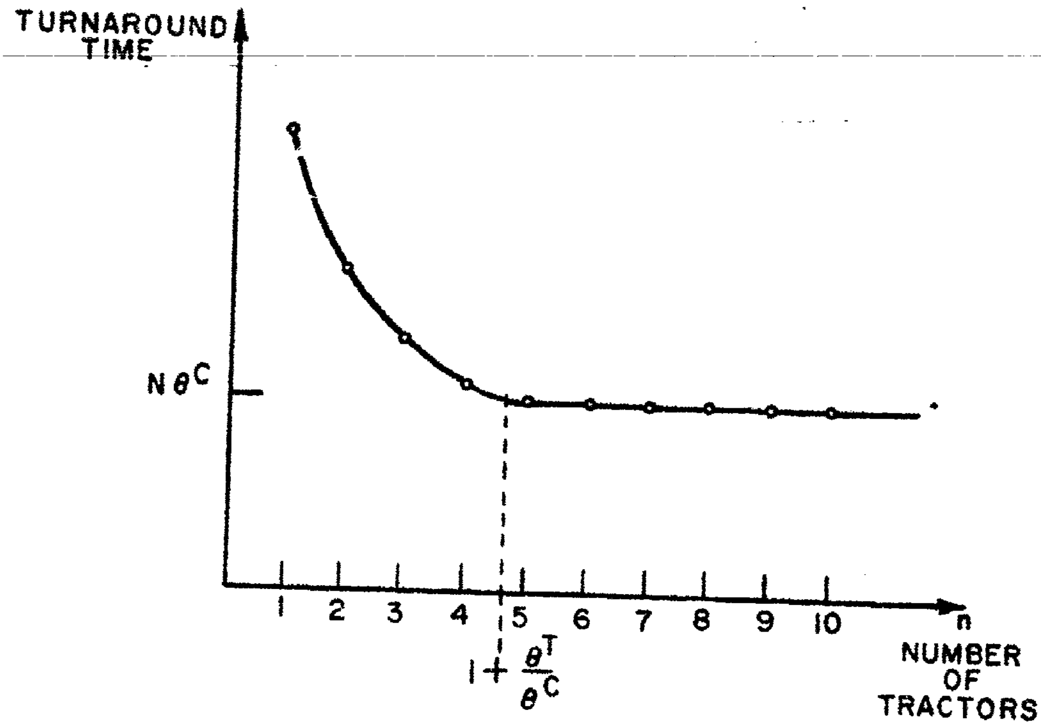


Fig. 7. Total ship turnaround time as a function of the number of tractors (deterministic case).

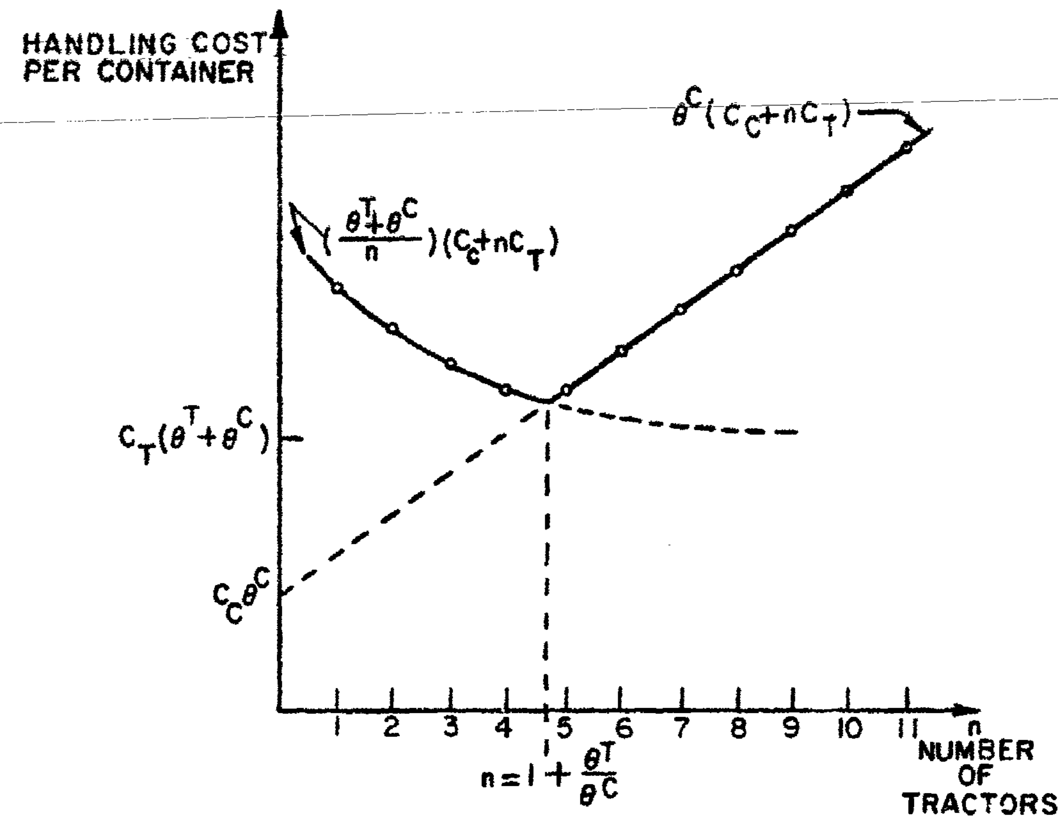


Fig. 8. Handling cost per container as a function of the number of tractors used (deterministic case).

It is evident that full utilization of the handling system in a rigidly sequenced operation is achieved by selecting the quantity $\theta = \frac{\theta^T}{\theta^c}$ as an integer; then, under optimum operating conditions, the cost of handling per container,

$$C_T\,\theta^T + (C_c + C_T)\,\theta^c \quad ,$$

is a linear function of the crane cycle time and the tractor travel time – the optimum value of n being $(\theta + 1)$, a quantity independent of the cost parameters. However, in general, such scheduling cannot be achieved even within a single loading-unloading operation of a vessel because the quantity θ^T is a function of the distance travelled by the tractor and, hence, a function of the location coordinate of the chassis in the marshalling area.

b. The Stochastic Model. In practice, a rigidly sequenced operation is impossible to achieve. The tractor travel time will vary between tractors and between travels while the cycle time operation of the crane will have inherent delays due to hatch removal, hatch replacement, latching operations, and movement of the cranes. We shall assume that arrival of the tractors to discharge berth is Poisson distributed with intensity of arrival λ; while the time for the crane to complete the one-on, one-off operation has an exponential distribution with parameter μ (note that $\theta_c = 1/\mu$ and $\theta_T = 1/\lambda$). We have considered, here, an extreme case of randomness; and, although the assumptions about the statistical distribution need verification, the results obtained can be used as bounds for the values of the variables involved.

The problem under study is similar to the classical repairman problem.[6] Let $P(m, t)$ be the probability that at time, t, there are m tractors waiting in line by the crane (tractors being serviced on a first come first served basis) $m \leq n$, then

$$P(0,t + dt) = P(0,t)(1 - n\lambda\, dt) + P(1,t)\,\mu\, dt\,[1 - (n - 1)\,\lambda\, dt] \qquad m = 0$$

$$\begin{aligned} P(m,t + dt) = {} & P(m,t)(1 - \mu dt)\,[1 - (n - m)\,\lambda\, dt] \\ & + P(m - 1,t)(1 - \mu dt)[n - (m - 1)\lambda\, dt] + P \\ & + P(m + 1,t)\,\mu\, dt\,[1 - \left(n - (m + 1)\right)\lambda\, dt] \qquad m = 1,2,\ldots,n \end{aligned}$$

These relations reduce to the following system of differential-difference equations:

$$\frac{dP(0,t)}{dt} = -\,n\lambda\, P(0,t) + \mu\, P(1,t) \qquad m = 0$$

[6] W. Feller, *Introduction to Probability Theory and its Applications*, Vol. 1, J. Wiley, New York, 1949.

$$\frac{dP(m,t)}{dt} = (n - m + 1)\lambda P(m - 1,t) - [\mu + (n - m)\lambda]\, P(n,t) + \mu P(m + 1,t)$$

$$m = 1,2, \ldots, n$$

In the steady state, let $P_m = P(m,t)$ $(m = o, 1, \ldots, n)$, then

$$0 = -n\lambda P_o + \mu P_1 \qquad m = 0$$

$$0 = (n - m + 1)\lambda P_{m-1} - (\mu + (n - m)\lambda)\, P_m + \mu P_{m+1} \qquad m = 1,2, \ldots, n$$

It is easy to verify that

$$P_m = \left(\frac{\lambda}{\mu}\right)^m \frac{n!}{(n - m)!} P_o \,, \qquad m = 1,2, \ldots, n$$

and $$P_o = \frac{1}{\left(\frac{\lambda}{\mu}\right)^n n! \sum_{m=o}^{m=n} \frac{(\mu/\lambda)^m}{m!}} = P_o(n)$$

P_o is the proportion of time the crane is idle. Since the expected number of containers handled per unit time by the crane while in operation (not idle) is μ, it follows that the expected number of containers handled per unit time is $(1 - P_o)\mu$.

(1) **Expected Turnaround Time (Fig. 9).** This is given by

$$\frac{N}{[1 - P_o(n)]\mu}$$

where N is the number of containers.

(2) **Optimum number of tractors in operation (Fig. 10).** Considering as the objective function the expected total cost of operation per container, C(n), we have

$$C(n) = \frac{C_c + n\, C_T}{\mu[1 - P_o(n)]}$$

The optimum number of tractors can then be determined by minimizing the function C(n).

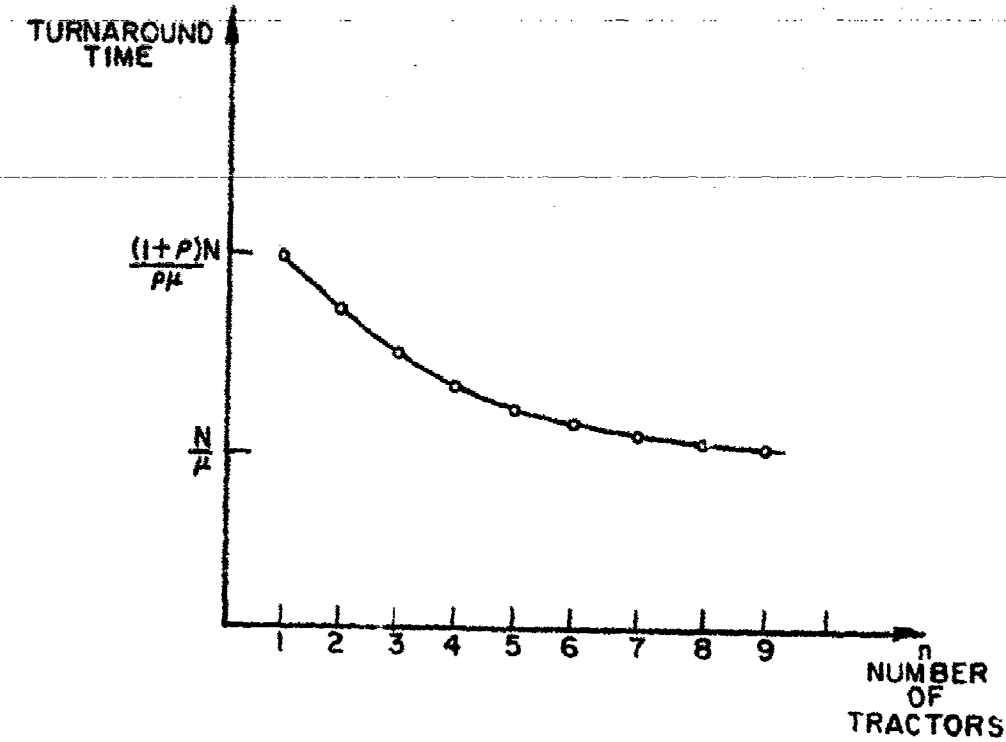


Fig. 9. Total ship turnaround time as a function of the number of tractors (stochastic case).

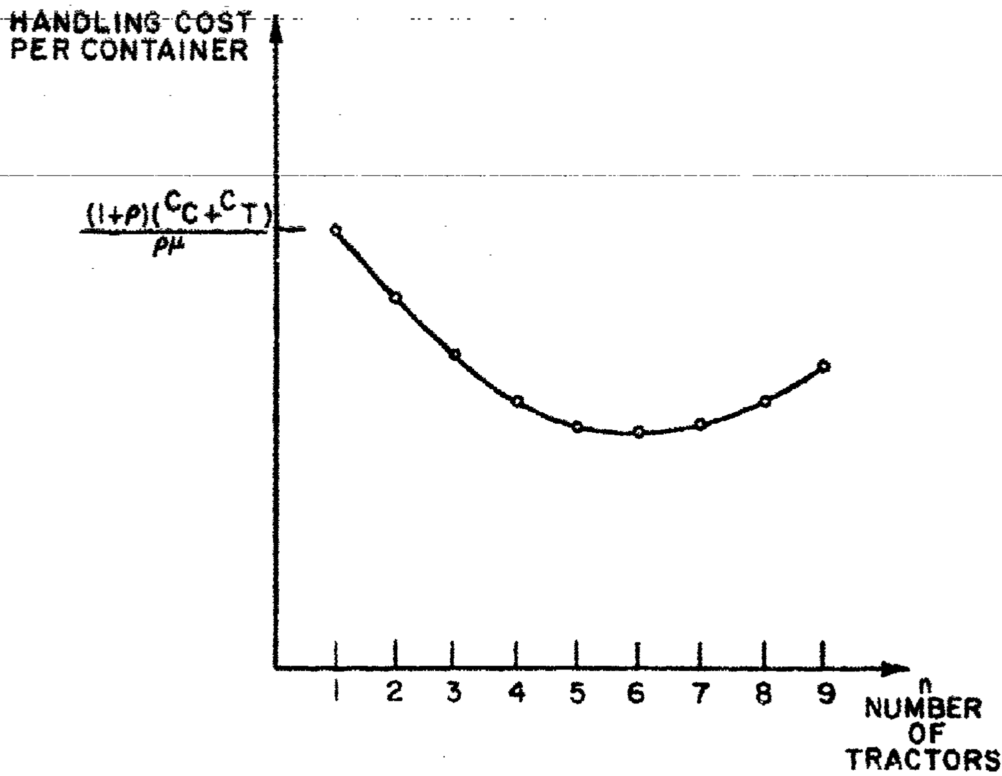


Fig. 10. Expected handling cost per container as a function of the number of tractors used (stochastic case).

12. Effect of Crane Breakdown on Turnaround Time. To this point, we have assumed that the equipments involved in the loading-unloading operation are not subject to breakdowns. The incorporation of this factor and its impact upon turnaround time is a significant problem that we shall now discuss. Since, in a Sea-Land terminal system, the crane is the bottleneck equipment, we shall study the effect of crane breakdown on turnaround time. Assume, first, that a single crane is operating. Let the capacity of the cargo vessel be N containers, and let T be the maximum permissible turnaround time. Under normal operation (no breakdowns), the ship turnaround time will be dictated by the number of containers loaded on the ship; however, in case of breakdown, the ship turnaround time might well be dictated by T (Fig. 11). Let the loading rate be v containers per unit time, and assume that breakdowns occur randomly in time according to a Poisson law with intensity λ. Let repair time be exponentially distributed with parameter μ. The situation is similar to the convoy retardation problem discussed in Section II. It is, thus, possible to obtain an expression for the expected turnaround time as a function of N, T, v, λ, and μ.

V. CONCLUSIONS

13. Conclusions. The mathematical analysis so far performed has been restricted to selected phases of the APSDS such as the movement of convoys across links and the operation at a node having a special configuration. The analysis is not claimed to be exhaustive; in fact, the solution to some of the mathematical problems encountered eludes us at present. Nevertheless, the results obtained so far constitute part of the basic inputs necessary to characterize the optimal configuration of the APSDS.

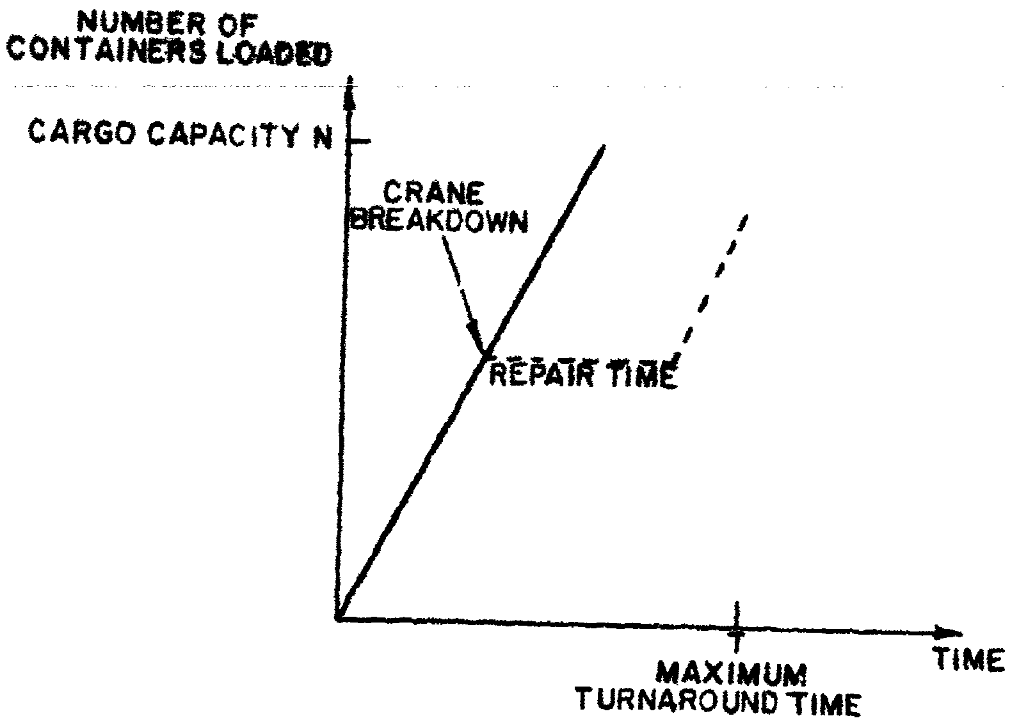
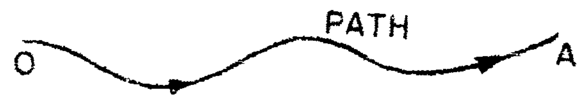


Fig. 11. Impact of crane breakdown upon the ship turnaround time.

APPENDIX A

CONCEPT OF MOBILITY OF AN AUTOMOTIVE EQUIPMENT SUBJECT TO BREAKDOWN AND REPAIRS

The mobility of an automotive piece of equipment operating under prescribed conditions can be defined as the probability that the equipment starting from a given origin, O, and moving along a given path will reach a predetermined destination, A, on this path on or before a specified time, T:



The quantity M(T) defined in Section II by expression (28) can thus be used to measure mobility. Note that M(T) incorporates the following four basic factors:

1. normal speed of motion, v
2. distance travelled, x
3. frequency of breakdown, λ
4. average repair period, $1/\lambda$

Consider two motor equipments E_1 and E_2 moving from O to A along OA at the same speed, v. Assume that E_1 and E_2 are subject to the same law of failures but that E_1 is supported by a better repair system than E_2; then, E_1 is more mobile than E_2.

APPENDIX B

PROPERTIES OF THE FUNCTIONS f(N) and g(N)

For convoy attrition Model 2, we attempt to characterize the properties of the functions $f(N)$ and $g(N)$ where $f(N)$ denotes the expected number of units in the convoy at a given time, t, and $g(N)$ represents the proportion of units lost. For notational simplicity, we shall carry on the derivations for the case when

$$\lambda(t) = \lambda \qquad \text{and} \qquad \mu(t) = \mu \, .$$

The derivations for the general case are identical and the same type of results are obtained.

Since $\quad f(N) = e^{-(\lambda+\mu)t} \, [NG(N) - \lambda t G(N-1)]$

Then
$$\begin{aligned} f(N+1) - f(N) &= e^{-(\lambda+\mu)t} \Big\{ (N+1)\, G(N+1) - \lambda t G(N) \\ &\quad - [NG(N) - \lambda t G(N-1)] \Big\} \\ &= e^{-(\lambda+\mu)t} \Big\{ (N+1) \left[G(N) + \frac{\lambda(t)^{N+1}}{(N+1)!} \right] - NG(N) \\ &\quad - \lambda t \, [G(N) - G(N-1)] \Big\} \\ &= e^{-(\lambda+\mu)t} \left[G(N+1) + \frac{N(\lambda t)^{N+1}}{(N+1)!} - \lambda t \, \frac{(\lambda t)^N}{N!} \right] \\ &= e^{-(\lambda+\mu)} \left[G(N+1) - \frac{(\lambda t)^{N+1}}{(N+1)!} \right] \\ &= e^{-(\lambda+\mu)t} \, G(N) > 0. \end{aligned}$$

Thus, $f(N)$ is an increasing function of N. The convexity of $f(N)$ can be determined by noting that

$$[f(N+2) - f(N+1)] - [f(N+1) - f(N)] = e^{-(\lambda+\mu)t} \, \frac{(\lambda t)^{N+1}}{(N+1)!} > 0 \quad .$$

Also, since $\quad \lim_{N \to \infty} \, [f(N+1) - f(N)] = e^{-\mu t} \, ,$

we have, for large N,

$$f(N) \approx A + N e^{-\mu t}$$

where $A < 0$ is a constant.

Next, we study the function $g(N)$:

$$g(N) = 1 - e^{-(\lambda+\mu)t}\left[G(N) - \frac{\lambda t}{N} G(N-1)\right]$$

$$g(N+1) - g(N) = 1 - e^{-(\lambda+\mu)t}\left[G(N+1) - \frac{\lambda t}{N+1} G(N)\right]$$

$$- \left\{1 - e^{-(\lambda+\mu)t}\left[G(N) - \frac{\lambda t}{N} G(N-1)\right]\right\}$$

$$= e^{-(\lambda+\mu)t}\left[-\frac{(\lambda t)^{N+1}}{(N+1)!} - \lambda t \frac{(N+1)G(N-1) - NG(N)}{N(N+1)}\right]$$

$$= -\frac{\lambda t e^{-(\lambda+\mu)t}}{N(N+1)}\left\{\frac{(\lambda t)^N}{(N-1)!} + NG(N-1) + G(N-1) - NG(N-1) - \frac{(\lambda t)^N}{(N-1)!}\right\}.$$

Finally,

$$g(N+1) - g(N) = -\frac{\lambda t e^{-(\lambda+\mu)t}}{N(N+1)} G(N-1) < 0 .$$

Therefore, $g(N)$ is a decreasing function of N. The convexity and/or concavity of $g(N)$ can be established as follows:

$$[g(N+2) - g(N+1)] - [g(N+1) - g(N)] = \lambda t\, e^{-(\lambda+\mu)t}\left[\frac{G(N)}{(N+1)(N+2)} - \frac{G(N-1)}{N(N+1)}\right]$$

$$= -\lambda t \frac{e^{-(\lambda+\mu)t}}{N+1}\left[2G(N-1) - \frac{(\lambda t)^N}{(N-1)!}\right]$$

Since this last expression can be either positive or negative depending on the specific values taken by λt and N, it is expected that, in general, $g(N)$ will not exhibit any particular convexity or concavity property.

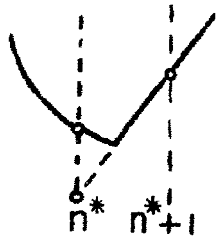
APPENDIX C

ANALYTIC PROPERTIES OF THE FUNCTION C(n)

Herein, we develop the conditions which will dictate the optimum number of tractors to be used when the quantity $\theta = \dfrac{\theta^T}{\theta^c}$ is not an integer (Case 2).

It is evident that the optimum number of tractors to use is either $n^* = [\theta+1]$ or $n^* = [\theta+2]$ or both, where $[x]$ is a symbol denoting the smallest integer in the real quantity, x. The appropriate value of n^* can be determined through considerations of the objective function which is taken to be the total cost for handling per unit container.

The convexity of the objective function guarantees the existences of an optimum n^*; however, in this case, a double optimal solution is possible. Further analysis to determine preferability of crane waiting rather than tractors (or vice versa) proceeds as follows:



Assume that at the optimal point, when $n = n^*$, the crane has to wait; then, $n^* < \theta + 1$. Clearly, if $(n^* + 1)$ tractors are used, the tractors will be waiting and the following inequalities should then hold:

$$\theta^c(C_c + n^* C_T) < (\theta^T + \theta^c)\left(\frac{C_c}{n^*} + C_T\right) < \theta^c\left[C_c + (n^* + 1)\, C_T\right]. \qquad (1)$$

The L.H.S. inequality yields

$$n^* < \theta + 1.$$

The R.H.S. inequality yields

$$(1 + \theta) < n^*\left(1 + \frac{1}{n^* + c}\right).$$

or $\quad n^{*2} + (c - \theta)\, n^* - c(1 + \theta) > 0$.

Since the corresponding quadratic equation has two real roots of opposite signs, this last inequality will hold for

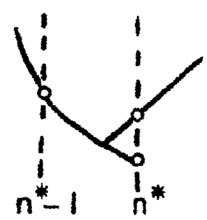
$$n^* > \frac{-(c-\theta) + \sqrt{(c+\theta)^2 + 4c}}{2}$$

It is easy to verify that the R.H.S. term of this inequality is greater than θ. Hence, the set of inequalities (1) can equivalently be written as:

$$\theta < \frac{-(c-\theta) + \sqrt{(c+\theta)^2 + 4c}}{2} < n^* < 1 + \theta. \qquad (2)$$

Assume, now, that at the optimal point, when $n = n^*$, the tractors have to wait; then,

$$n^* > 1 + \theta .$$



Since, if $(n^* - 1)$ tractors are used, the crane will have a waiting time, it follows that:

$$\frac{\theta^T + \theta^c}{n}\,(C_c + n^* C_T) < \theta^c\,(C_c + n^* C_T) < \frac{\theta^T + \theta^c}{n^* - 1}\,(C_c + (n^* - 1) C_T). \qquad (3)$$

The L.H.S. inequality yields

$$n^* > 1 + \theta .$$

The R.H.S. inequality yields

$$(n^* - 1)(c + n^*) < (1 + \theta)(c + n^* - 1),$$

$$n^{*2} + (c - 2 - \theta)\, n^* - 2c + 1 - \theta c + \theta < 0 .$$

The corresponding quadratic equation can be shown to have two real roots given by

$$\frac{-(c-\theta) \pm \sqrt{(c+\theta)^2 + 4c}}{2} + 1$$

Consequently,

$$1 + \frac{-(c-\theta) - \sqrt{(c+\theta)^2 + 4c}}{2} < n^* < 1 + \frac{-(c-\theta) + \sqrt{(c+\theta)^2 + 4c}}{2}$$

It can be verified that the left-most quantity of the above expression is less than $1 + \theta$; hence, the set of inequalities (3) can equivalently be written as

$$(1+\theta) < n^* < 1 + \frac{-(c-\theta) + \sqrt{(c+\theta)^2 + 4c}}{2} \quad .$$